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PLASTIC DEFORMATION IN BEAMS UNDER SYMMETRIC DYNAMIC LOADS

by

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Abstract

Plastic deformations in beams caused by dynamic loads distributed over finite lengths are considered in this paper. The analysis is based on the assumption of "plastic-rigid" behavior so that the results should be valid when plastic deformations are sufficiently large. The final deformations for distributed loads are found not to differ much qualitatively from those to be expected for concentrated loads. However, the magnitude of the deformation decreases rapidly as the length of the loaded area increases from zero, for given load pulses. Thus the simplifying assumption of a mathematically concentrated load may cause a considerable over-estimate of the final deformation.

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1. Introduction.

In a recent paper [1]<sup>1</sup> a method was described for calculating the permanent deformation of a bar of ductile metal subjected to a transverse impact load concentrated at the mid-point of the bar. The basis of the method, originally suggested by Prager, is that elastic strains are neglected entirely, the bar being assumed to be rigid except at cross-sections where the limit, or fully plastic, moment  $M_0$  is maintained. At such sections it is assumed that plastic strains of indefinitely large magnitude can occur. Since discontinuities in slope angle can take place at such sections, while the bending moment remains essentially constant there, these sections are conveniently termed "plastic hinges". This concept has been found to have valuable applications in predicting static loads which produce failure of the "plastic collapse" type in civil engineering structures. References [2, 3] may be consulted for descriptions of these applications, and for details concerning the significance and calculation of the limit moment magnitude.

Whether or not the concept of localized deformation at plastic hinge sections can be usefully applied in problems of

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1. Numbers in square brackets refer to the Bibliography at the end of the paper.

dynamic loading depends largely on how much the energy actually absorbed during plastic deformation under a given dynamic load exceeds the maximum elastic strain energy which could be stored in the structure. A criterion based on this hypothesis was presented in [1]. Experimental investigations designed to test the predictions of this so-called "plastic-rigid" type of dynamic analysis will be highly desirable, and it is hoped that the results of the present paper will be helpful in planning and interpreting such experimental work.

It will be particularly desirable, in connection with experiments and with practical applications in general, to know how certain load characteristics may be expected to influence the results. In a previous paper [4], the influence of characteristics of force pulses was investigated, the force being supposed mathematically concentrated at the mid-section of the beam; as in [1], the case of a uniform beam with free ends was considered. In [4] a simple empirical formula was suggested by means of which the final central angle of deformation produced by a concentrated force could be estimated from knowledge of the total impulse and peak force value; the shape of the force-time curve was found to be relatively unimportant.

The present paper is concerned with the differences from the results of [4] to be expected when the load is distributed over a finite length of the beam rather than concentrated at the mid-section. In order to obtain both qualitative and quantitative comparisons as simply as possible, the only form of load distribution curve used is that indicated in Fig. 1.

The total load  $P$  is uniformly distributed over a length  $2e$  where  $e \leq l$ . Thus the parameter specifying the distribution of the load is the ratio  $\frac{e}{l} \equiv k$ .

A preliminary discussion of this problem has been given by Salvadori and DiMaggio [5]. They used a smoother but more complicated load distribution function, the load intensity being written as:

$$p(x) = \frac{P}{2l[1 - e^{-c(1 + \frac{e}{2})}]} \left[ \frac{c}{2} \left( 1 + c \frac{x}{l} \right) e^{-c \frac{x}{l}} \right], \quad (1)$$

where  $P$  is the total load,  $x$  is distance from midsection, and  $c$  is a concentration parameter which may vary from zero (corresponding to a uniform distribution over the entire beam), to infinity (corresponding to concentration of the load at the midsection). The authors carried their study only up to the point of showing that a sufficiently "spread out" load, characterized by a value of the parameter  $c$  less than 3.45, would apparently introduce a qualitative difference in the behavior of the beam. This apparent difference will be discussed later in the present paper. With a simpler form of load distribution, as used in this paper, it is not difficult to carry the investigation further, and to obtain quantitative comparisons between plastic deformations occurring with various degrees of localization. These quantitative comparisons and more complete qualitative comparisons between final plastic deformations are the main purpose of the present paper. The main practical result is that a small degree of spreading of the load over a finite length of the beam leads to a considerable reduction of the total deformation at the midsection. Qualitatively, however, the present

analysis shows that the final shape of the deformed beam will not differ radically from that predicted by analyses assuming the force to be concentrated at the midsection.

## 2. Analysis of Earlier Stages of Deformation.

Taking the load to be uniformly distributed over the length  $2e$  as shown in Fig. 1, we consider the successive phases of motion and deformation which can occur. We suppose the total load  $P$  increases steadily up to a maximum value and then decreases to zero, so that force pulses of the general shape shown in Fig. 2 are considered.

Under these conditions, as the force increases from zero its first effect is to cause a translation of the bar parallel to itself. Then all points of the bar have the acceleration:  $a = \frac{P}{2m\ell}$ , where  $m$  is the mass per unit length. This initial translation is termed the "first phase" of the motion. The constant acceleration has the effect of a uniformly distributed loading, and the bending moment at any section  $x$  [positive bending moments defined as in Fig. (4b)] is given by:

$$\begin{aligned} x > e: \quad M(x) &= -\frac{m(\ell - x)^2}{2} a \\ x < e: \quad M(x) &= -\frac{m(\ell - x)^2}{2} a + \frac{P}{2e} \frac{(e - x)^2}{2}, \end{aligned} \quad (2)$$

where  $x$  is measured from the middle of the beam. The bending moment in this phase has a minimum at  $x = 0$  given by

$$M(0) = -\frac{m\ell^2}{2} a + \frac{Pe}{4} = -\frac{P\ell}{4} + \frac{Pe}{4}. \quad (2a)$$

This first phase ends when the central moment reaches the value  $-M_0$ , corresponding to a load  $P_I$  given by

$$\frac{P_I \ell}{M_0} = \frac{4}{1 - \frac{e}{\ell}} : \quad \text{or} \quad \mu_I = \frac{4}{1 - k} \quad (3)$$

where, for convenience, we define  $\frac{P \ell}{M_0} = \mu$  and  $\frac{e}{\ell} = k$ .

In the "second phase" of the motion, the two halves remain rigid but rotate with respect to each other, as in Fig. (3a). Let  $a_0$  denote the linear acceleration of the center, and  $\alpha_0 = \frac{d^2 \theta_0}{dt^2}$  denote the angular acceleration of the right hand half (positive clockwise). Assuming  $\theta_0$  to remain small the equations of motion are:

$$\frac{P}{2} = m \ell (a_0 - \frac{1}{2} \ell \alpha_0); \quad \frac{P(\frac{\ell}{2} - \frac{e}{2})}{2} - M_0 = \frac{m \ell^3 \alpha_0}{12} . \quad (4)$$

Solving these for  $a_0$  and  $\alpha_0$ , we obtain:

$$\begin{aligned} a_0 &= \frac{M_0}{m \ell^2} \left[ \mu \left( 2 - \frac{3}{2} k \right) - 6 \right] \\ \ell \alpha_0 &= \frac{M_0}{m \ell^2} [3\mu(1 - k) - 12] . \end{aligned} \quad (5)$$

This second phase continues until the moment at some additional cross-section reaches the magnitude  $M_0$ .

To see how this may occur, it is helpful to refer to typical diagrams of effective load, shear force, and bending moment in the right hand half of the beam. The effective load intensity  $q$ , with sign defined as in Fig. 4(b), can be regarded as made up of three parts, as shown in Fig. 4(a); these are the intensity  $\frac{P}{2e} = \frac{M_0}{\ell^2} \frac{\mu}{2k}$  of the given load distributed over the length  $e$ , the uniform load intensity  $-m a(\frac{\ell}{2}) = -\frac{M_0}{\ell^2} \frac{\mu}{2}$  corresponding to the acceleration  $a(\frac{\ell}{2})$  of the center of gravity of the half beam, and finally the linearly varying effective load intensity



corresponding to the angular acceleration  $\alpha_0$  of the half-beam, whose maximum value is  $m\alpha_0 \frac{l}{2} = \frac{M_0}{l^2} [\frac{3}{2} \mu(1 - k) - 6]$ .  $q$ ,  $V$ ,  $M$ , curves are shown in Figs. (5a), (5b), and (5c).

Two limiting conditions are evident from the diagrams of Fig. (5). In the first place, unless  $q(l)$  is positive, the  $V$  curve cannot cut the axis in the unloaded position  $x > e$ , and hence there can be no positive moment in the beam. Hence, in order for it to be possible to develop a plastic hinge at a point where the maximum moment reaches the value  $+M_0$ , we must have  $\mu$  and  $k$  related so that

$$\frac{3}{2} \mu (1 - k) - 6 - \frac{\mu}{2} > 0 \quad (6)$$

or

$$\mu(1 - \frac{3}{2} k) > 6. \quad (6a)$$

Secondly, at the left hand end of the half beam,  $q(0)$  must remain positive; if it did not  $V$  would have negative values in the neighborhood of  $x = 0$ , and hence  $M(x)$  would decrease below  $-M_0$  in this region. This requires the following relation between  $\mu$  and  $k$ :

$$\frac{\mu}{2k} - \frac{3}{2} \mu (1 - k) + 6 - \frac{\mu}{2} > 0 \quad (7)$$

or

$$\mu(3k - 1)(1 - k) < 12k. \quad (7a)$$

It will then be evident that the second phase may be terminated in some cases because there is a positive moment in the interior of the half-beam (as in Fig. 5(c)) which reaches the value  $M_0$  at a sufficiently large value of the load; and in other cases because the moment in the neighborhood of the mid-

section reaches the negative limit moment value  $-M_0$  over a finite length of the beam, as in Fig. 3(c).

The first case, occurrence of a "positive" plastic hinge in the interior, is possible only if the inequality (6) is satisfied. From 6(a), therefore, it is evident that in order for this new hinge to form in the interior of each half-beam it is necessary to have:

$$k < \frac{2}{3}. \quad (8)$$

The second case, of spreading of the central hinge, can occur only if inequality (7) is not satisfied, and hence only if

$$k > \frac{1}{3} \quad (9)$$

since inequality (7a) will be satisfied, regardless of  $\mu$ , if  $k \leq 1/3$ .

In the case of a concentrated force, [1, 4], the second phase ends when the positive bending moment reaches the value  $M_0$ , and this always occurs when the load parameter  $\mu$  reaches the value  $\mu_{II} = 22.89$ . The considerations of the preceding paragraph show that when the load is sufficiently spread out ( $k \geq 2/3$ ), the formation of interior positive hinges will not take place in this manner. This is the main qualitative difference caused by loading over a finite length of the beam. Instead of new out-board hinges making their appearance, for sufficiently large  $k$  values the new effect is the spreading of the central hinge, which in the earlier part of the motion is always concentrated (according to the basic hypotheses of the method) at a single section.

We now put in more precise terms the conditions for (1) occurrence of a positive plastic hinge in the interior of the beam, and (2) spreading of the central negative plastic hinge.

(1) Occurrence of lateral plastic hinge:

For  $e \leq x \leq l$ , the bending moment is given by

$$M(x) = \frac{P}{2} \left(x - \frac{e}{2}\right) - M_0 - \frac{mx^2 a_0}{2} + \frac{mx^3 \alpha_0}{6}. \quad (10)$$

This moment will have a stationary maximum at a point which can be found by solving the equation

$$V = \frac{dM}{dx} = \frac{P}{2} - mxa_0 + \frac{mx^2 \alpha_0}{2} = 0. \quad (11)$$

Using Eqs. (5) this yields

$$\frac{x}{l} = \frac{\mu}{3\mu(1-k) - 12}. \quad (12)$$

This value of  $x$  is put in Eq. (10) to give the expression for the maximum moment. Then the value of  $\mu$  for which the second phase ends is found by setting the resulting expression equal to  $M_0$ . The following cubic equation in  $\mu_{II}$  and  $k$  is obtained:

$$(3k-2)^2 \mu_{II}^3 + 36(12k^2-18k+7) \mu_{II}^2 + 423(5k-4) \mu_{II} + 3456 = 0. \quad (13)$$

For any particular value of  $k$ , the real root of Eq. (13) then gives the value of  $\mu$  for which the lateral hinges first appear, on the assumption that the kinematic picture of Fig. (3b) is correct.

(2) The central hinge must begin to spread if

$$\left. \frac{d^2 M}{dx^2} \right|_{x=0} = \left. \frac{dV}{dx} \right|_{x=0} = q(0) = 0. \quad (14)$$

The critical relation is therefore obtained by replacing the inequality in (7a) by an equality, which yields

$$\mu_{II} = \frac{12k}{(3k - 1)(1 - k)} . \quad (15)$$

The curves of  $\mu$  as a function of  $k$  according to Eqs. (13) and (15) are plotted in Fig. 6. It is found that the positive branch of the curve from Eq. (15) crosses that from Eq. (13) at  $k = 0.361$ ,  $\mu_{II} = 81.4$ . We conclude, therefore as follows:

(1) If  $k < 0.361$  the second phase ends with the formation of outboard hinges. The initial location of these hinges will be found from Eq. (12), and the value of the load parameter  $\mu$  at which they appear is the real root of Eq. (13). Plastic regions appear in the manner indicated in Fig. (3b).

(2) If  $k > 0.361$  the termination of the second phase will be due to the spreading of the central plastic hinge into a finite region in which the moment has the constant value  $-M_0$ . The load parameter at which this effect begins is given by Eq. (14), and the corresponding type of deformation is indicated in Fig. (3c).

This critical value of  $k = 0.361$  corresponds to the critical value  $c = 3.45$  found by Salvadori and DiMaggio for the smooth load distribution curve assumed by them, Eq. (1). However, they refer to "splitting of the central hinge" as the effect resulting when Eq. (14) is satisfied before the lateral hinges form. They state that when Eq. (14) is satisfied, "the moment  $M$  becomes minimum at  $x = 0$  and the central hinge splits into two symmetrical hinges which wander outward, plasticizing a central portion of the beam". This suggests that two distinct hinges are

present in place of the original central hinge, with the portion of the beam between them subjected to a moment less in magnitude than  $M_0$ , and hence moving as a rigid body. This could only occur, however, if the load intensity has a relative minimum at the center, which is not the case in the distribution curve assumed by them. In the usual type of load distribution met with in impact problems in which the load intensity is either constant or a relative maximum at the midpoint, the effect is not a "splitting" but a spreading of the central hinge. In other words, unless the load intensity increases (near  $x = 0$ ) with distance from the midpoint, the effect is the development of a "hinge" of finite length, with the moment constant at the value  $-M_0$  in a finite region which increases in size as the load increases.

For the analysis of the motion after the close of the second phase we assume first that the load distribution parameter  $k = e/\ell$  is less than 0.361 and then that  $k$  exceeds 0.361.

### 3. Analysis for $k < 0.361$ .

In this case the lateral hinges appear while the central hinge is still localized at the center section. The appropriate free body diagrams for the right hand half are shown in Fig. (7), the distance of the lateral hinge from the center being denoted by  $x_h$ . It is assumed that  $x_h > e$ . The equations of motion, as in [1, 4] are

$$\frac{P}{2} = mx_h(a_0 - \frac{1}{2} x_h \alpha_0) \quad (16a)$$

$$\frac{P}{2} \left( \frac{x_h}{2} - \frac{e}{2} \right) - 2M_0 = \frac{mx_h^3 \alpha_0}{12} \quad (16b)$$

$$0 = m(\ell - x_h) \left[ a_1 + \frac{1}{2}(\ell - x_h) \alpha_1 \right] \quad (17a)$$

$$M_0 = \frac{1}{12} m(\ell - x_h)^3 a_1. \quad (17b)$$

The fifth equation relating the five unknowns  $a_0$ ,  $a_1$ ,  $\alpha_0$ ,  $\alpha_1$  and  $x_h$  is that which expresses the discontinuity of linear acceleration at the hinge location  $x_h$ , which is in general a function of time. This is (See [1, 4])

$$a_0 - x_h \alpha_0 - [a_1 + \alpha_1(\ell - x_h)] = \frac{dx_h}{dt}(\omega_0 - \omega_1) \quad (18)$$

where  $\omega_0$ ,  $\omega_1$  are the angular velocities of the segments to the left and to the right, respectively, of the hinge, i.e.,

$$\alpha_0 = \frac{d\omega_0}{dt}, \quad \alpha_1 = \frac{d\omega_1}{dt}. \quad (19)$$

Equations (16) - (18) yield the following system of equations:

$$\alpha_0 = \frac{d\omega_0}{dt} = \frac{3M_0}{m\ell^3 \xi^3} [\mu(\xi - k) - 8] \quad (20a)$$

$$\alpha_1 = \frac{d\omega_1}{dt} = \frac{12M_0}{m\ell^3(1 - \xi)^3} \quad (20b)$$

$$a_0 = \frac{M_0}{2m\ell^2 \xi^2} [\mu(4\xi - 3k) - 24] \quad (20c)$$

$$a_1 = - \frac{6M_0}{m\ell^2(1 - \xi)^2} \quad (20d)$$

$$\frac{M_0}{m\ell^3} \left[ \frac{\mu}{\xi} - \frac{3}{2} \frac{\mu k}{\xi^2} - \frac{12}{\xi^2} + \frac{6}{(1 - \xi)^2} \right] = - \frac{d\xi}{dt}(\omega_0 - \omega_1) \quad (20e)$$

where

$$\xi = \frac{x_h}{\ell}. \quad (21)$$

These equations hold so long as the bending moment has a magnitude less than  $M_0$  everywhere except at the sections  $x = 0$  and  $x = x_h$ . The effective load, shear, and moment diagrams for the segments to the left and right of the moving hinge, Fig. (8)

indicate that the critical condition will be a spreading of the central hinge. This will occur when the following condition is satisfied

$$\left. \frac{d^2 M}{dx^2} \right|_{x=0} = \left. \frac{dV}{dx} \right|_{x=0} = q(0) = 0. \quad (22)$$

From Eq. (16) this leads to the condition

$$\mu = \frac{24k}{(\xi - k)(3k - \xi)}. \quad (23)$$

The use of the limiting condition requires that Eqs. (20) first be solved, yielding  $\xi$ , (among other quantities), as a function of  $\mu$ ; these solutions will be valid as long as  $\mu$  is less than the value computed from Eq. (23).

If any given finite force pulse is replaced by a rectangular pulse of the same peak force and the same area, i.e., total impulse, the final deformations computed for the rectangular pulse will exceed the true ones for the original pulse, but for many purposes the results will be useful estimates (e.g., in error by 15 or 20 percent), see [4]. Therefore to obtain quantitative comparisons of total deformations with various values of  $k$  we have assumed a rectangular force pulse shape, Fig. (2b). Since the details of the method of analysis exactly parallel those in [4], only the final results are shown in Fig. (12) where curves of the deformation parameters  $m\ell^3\theta_0/M_0T^2$  are plotted as functions of  $\mu_m \equiv P_m\ell/M_0$  for a series of values of the distribution parameter  $k$ . These results and others are discussed further in the final section of the paper.

#### 4. Analysis for $k > 0.361$ .

In this case the central hinge begins to spread before lateral hinges appear. At time  $t$  the central hinge will have spread out a distance  $d$  from the center; in the region  $-d \leq x \leq d$  the moment is constant at the value  $-M_0$ . The equations of motion for the segment of length  $\ell - d$  are obtained from the free body diagrams of Fig. (9), and are found to be:

$$\frac{P}{2e}(e - d) = m(\ell - d)[a'_0 - \frac{1}{2}(\ell - d)\alpha'_0] \quad (24a)$$

$$\frac{P}{2e}(e - d)\left[\frac{\ell - d}{2} - \frac{e - d}{2}\right] - M_0 = \frac{1}{12}m(\ell - d)^3\alpha'_0 \quad (24b)$$

where  $a'_0$  is the acceleration at  $x = d$  and  $\alpha'_0$  is the angular acceleration of the segment of length  $\ell - d$ .

Equations (24) yield expressions for the accelerations as follows:

$$a'_0 = \frac{M_0}{m\ell^2(1 - \lambda)^2} \left[ \frac{\mu}{2k}(k - \lambda)(4 - \lambda - 3k) - 6 \right] \quad (25a)$$

$$\alpha'_0 = \frac{12M_0}{m\ell^3(1 - \lambda)^3} \left[ \frac{\mu}{4k}(k - \lambda)(1 - k) - 1 \right] \quad (25b)$$

where

$$\lambda \equiv \frac{d}{\ell}. \quad (25c)$$

To determine  $d$ , we use the fact that the central hinge spreads at a rate just sufficient to prevent the moment near  $x = d$  from decreasing below the value  $-M_0$ . The condition is that

$$\left. \frac{d^2 M}{dx^2} \right|_{x=d} = \left. \frac{dV}{dx} \right|_{x=d} = q(d) = \frac{P}{2e} - ma'_0 = 0. \quad (26)$$

Using Eq. (25a) and simplifying, it is found that  $\lambda$  is the following function of  $\mu$  and  $k$ :



$$\lambda = \frac{\mu(3k - 1)(1 - k) - 12k}{2\mu(1 - k)} . \quad (27)$$

Plots of  $\mu$  vs.  $\lambda$  for various values of  $k$  are given in Fig. (10). Thus for  $k > 0.361$  the critical portion of the beam ( $-x \leq \lambda\ell \leq x$ ) has the constant moment  $-M_0$ , so that the accelerations of all elements are proportional to the local load intensity  $P/2e$ . The two segments of length  $\ell - d = \ell(1 - \lambda)$  rotate as if rigid bars hinged at the sections  $x = \pm d$ , Fig. (3c). So long as the moment in the outer segments remains less than  $M_0$  in magnitude, this is the correct picture. The load magnitude at which this picture ceases to be the correct one is that at which new "positive" hinges form in the interior of the outer segments. The maximum bending moment in these segments can be investigated in the same manner as was done for the case of a localized central hinge. The following equation was obtained by setting the maximum bending moment in one segment  $d \leq x \leq \ell$  equal to  $M_0$ . It is analogous to Eq. (16) but depends now upon  $\lambda = d/\ell$  in addition to  $k$ .

$$2 = \frac{\mu^2(k - \lambda)^2(1 - \lambda)^2[\mu(k - \lambda)(2 + \lambda - 3k) - 12k]}{4k[3\mu(k - \lambda)(1 - k) - 12k]^2} - \frac{\mu(k - \lambda)^2}{4k} + \frac{\mu^3(k - \lambda)^3(1 - \lambda)^3}{6k[3\mu(k - \lambda)(1 - k) - 12k]^3} . \quad (28)$$

To determine the load at which the additional hinges first appear, the above equation must be solved simultaneously with Eq. (27). Alternatively, for any chosen value of  $k$  ( $> 0.361$ ) plots of  $\mu$  vs.  $\lambda$  from Eq. (28) and Eq. (27) can be drawn, and the intersection of the two curves yields the initial value of  $\mu$  in question. As

one example, the case  $k = 0.5$  has been worked out. In this case, when the two equations are combined we obtain

$$\mu^4 - 192\mu^3 + 1152\mu^2 - 110592 = 0. \quad (29)$$

This has solution  $\mu = 186$ , corresponding to  $\lambda = 0.242$ ; the new hinge appears at  $x = 0.660\lambda$ .

## 5. Numerical Results and Conclusions.

The type of load distribution considered here, with uniform load intensity over part of the beam and zero load over the rest, is intended less as a realistic loading than as the simplest type of distributed load to handle analytically. It seems intuitively clear that the qualitative results deduced from this simple case will apply to general cases of loads spread over finite areas. The main qualitative result is, in fact, that the general picture of the final deformation is exactly the same as that in the case of concentrated loading unless the length over which the load is applied exceeds about a third of the total length of the beam. Only when  $k$  exceeds the value 0.361 does a qualitative difference appear. This consists of a spreading of the central hinge, so that instead of a plastic hinge localized at the midsection there is a finite length of the beam in which the bending moment has the magnitude  $M_0$ . However in real beams a hinge cannot be localized at a single cross-section in any case since this would imply infinite strains; the plastic region is always spread over a finite region even when the load is strongly localized. Thus the final plastic deformation of a beam subjected to a sufficiently large distributed load will not in

any case differ radically from that predicted by an analysis using a mathematically concentrated load.

As far as quantitative comparisons are concerned it is of particular interest to investigate how a small degree of load spreading modifies the magnitude of the final deformation predicted by the simpler analysis for a concentrated force, since in practice loads are always applied over finite areas.

To provide information of this kind calculations have been made of the final central angle  $\theta_0$  produced by a series of loads with  $k$  values ranging from zero to 0.30. The force-time curve was taken, for maximum simplicity, to have a square-wave shape, as in Fig. 2(b). For this type of force pulse the lateral hinges move instantaneously to positions which they maintain so long as the load remains constant. The lateral hinge coordinate  $\xi l$  is found as the solution of the following equation:

$$\frac{\mu}{\xi} - \frac{3\mu k}{2\xi^2} - \frac{12}{\xi^2} + \frac{6}{(1-\xi)^2} = 0. \quad (30)$$

This is Eq. (20e) with the right hand side set equal to zero. Figure 11 shows curves of  $\xi$  plotted against  $\mu$  for values of  $k$  ranging from zero up to 0.30. The calculation of the further increments of deformation which occur after the load drops to zero are made in the manner described in [4].

Figure 12 shows curves of the dimensionless parameters  $m l^3 \theta_0 / M_0 T^2$  plotted as a function of the dimensionless load parameter  $\mu_m = P_m l / M_0$ , for  $k$  equal to 0, 0.10, 0.20, and 0.30. Figure 13 shows curves of  $m l^3 \theta_0 / M_0 T^2$  plotted against  $k$  for various values of  $\mu_m$ . These curves show that when the maximum load is large

(e.g.,  $\mu_m > 40$ ) the deformation falls off very rapidly. For example, with  $\mu_m = 40$ , the deformation parameter drops from about 800 for a concentrated load ( $k = 0$ ) to about 600 for  $k = 0.05$ . This apparently large reduction caused by the distribution of the load over a finite length may be somewhat exaggerated, as compared with practical cases, because of the special conditions of the present calculation, i.e., the assumption of a rectangular force pulse and of a constant load intensity over the loaded segment of the beam. However the results indicate that the assumption of a mathematically concentrated load may lead to an appreciable overestimate of the deformation, since the actual load is spread over a small but finite length of the beam. This result should be of significance in evaluating experimental results. It should perhaps also be noted that strain-hardening will cause a further reduction in the permanent deformation; this effect will presumably be more important the higher the degree of concentration of the load.

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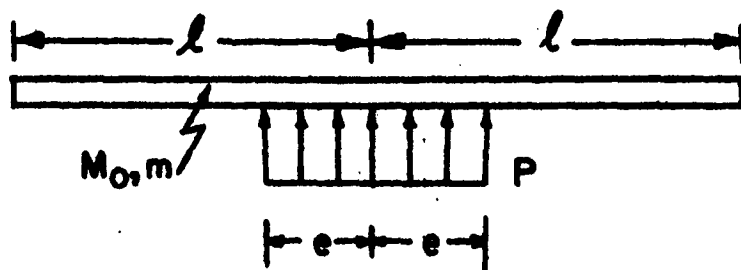


Fig. 1

Type of load distribution assumed in this paper

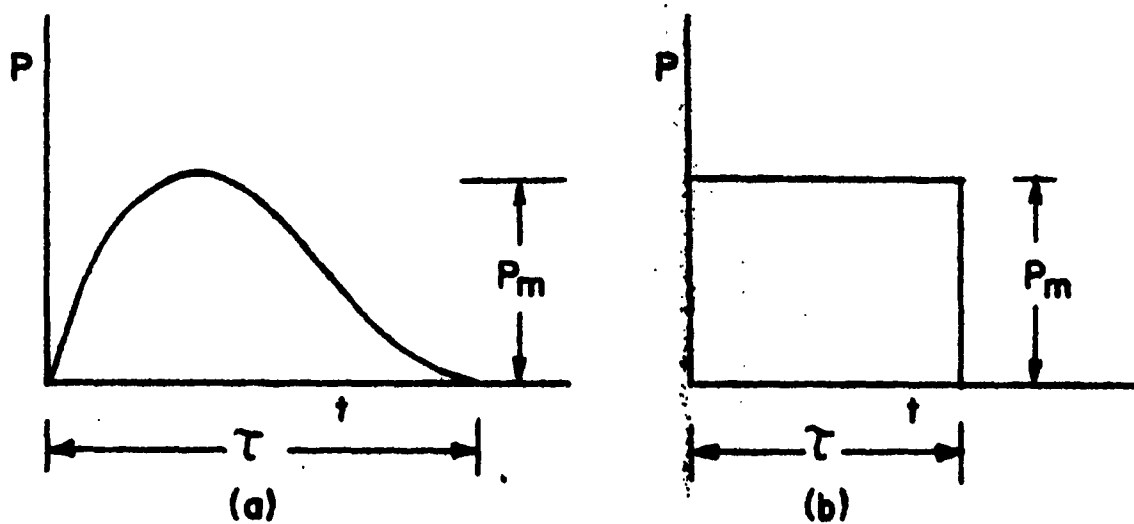


Fig. 2

Typical force pulse loadings

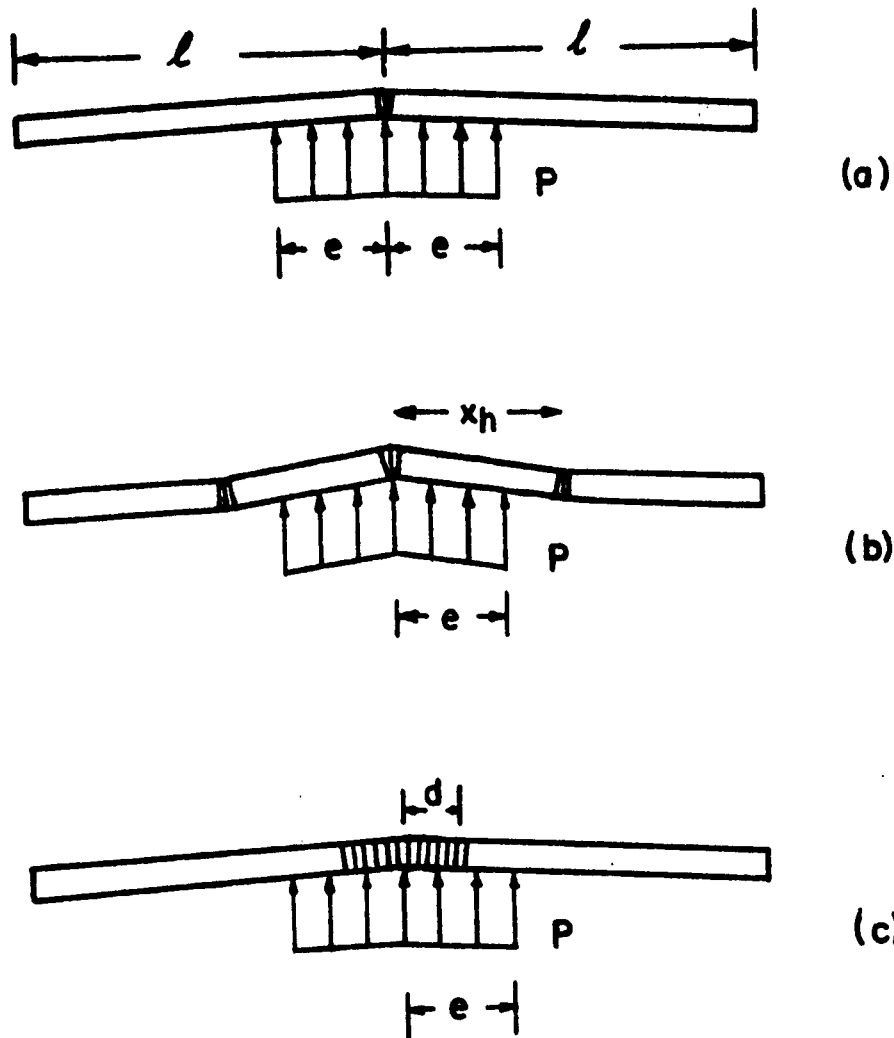


Fig. 3

Possible types of deformation: (a) Second phase; plastic hinge at midpoint only; (b) Third phase with additional hinges at interior points in each half-beam; (c) Third phase with development of finite central plastic region.

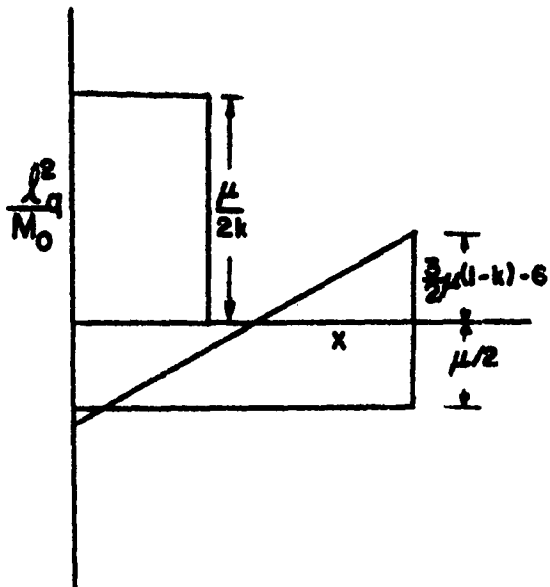


Fig. 4(a)

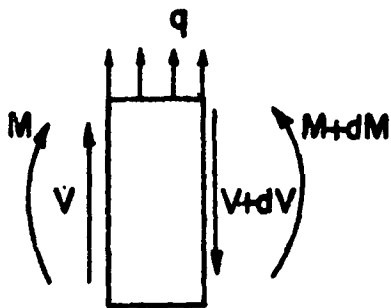


Fig. 4(b)

Effective load components  
in the second phase

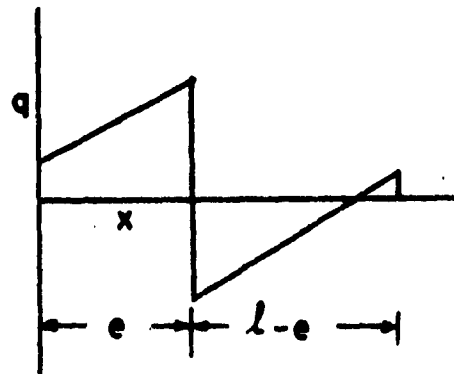


Fig. 5(a)

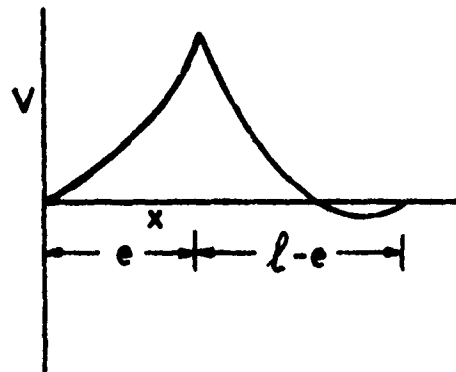


Fig. 5(b)

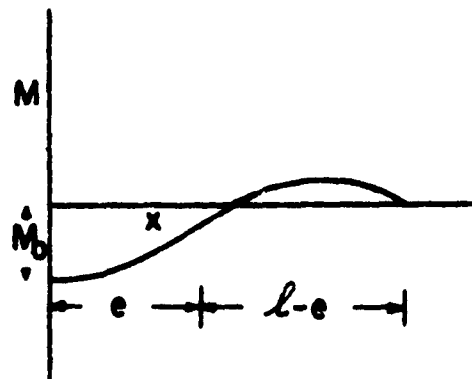


Fig. 5(c)

Typical diagrams of effective  
load intensity and associated  
shear and bending moment dia-  
grams in second phase.



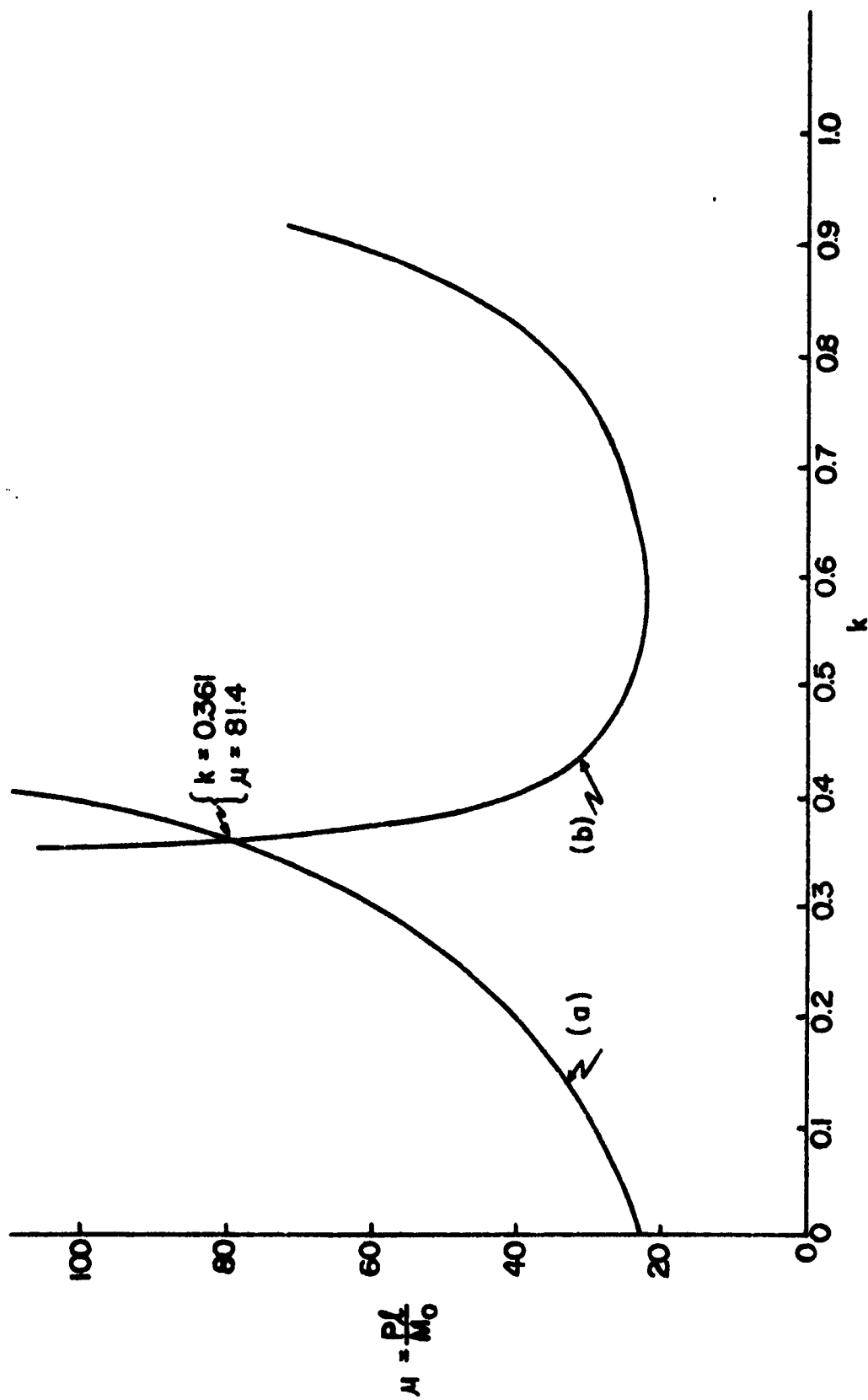


Fig. 6

Plots of  $\mu$  at which the second phase ends as junction of  $k = e/l$  corresponding to (a) appearance of lateral "positive" plastic hinges; and (b) spreading of the central hinge

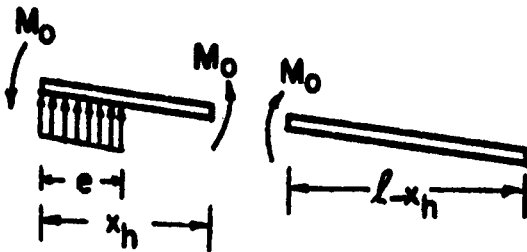


Fig. 7

Free body diagrams for motion in third phase with concentrated central hinge

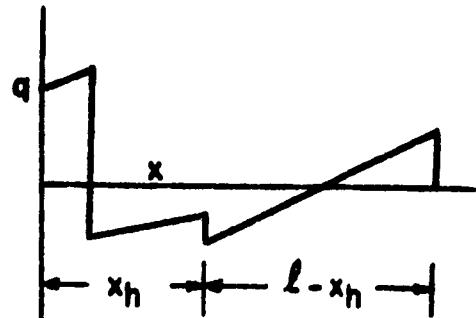


Fig. 8(a)

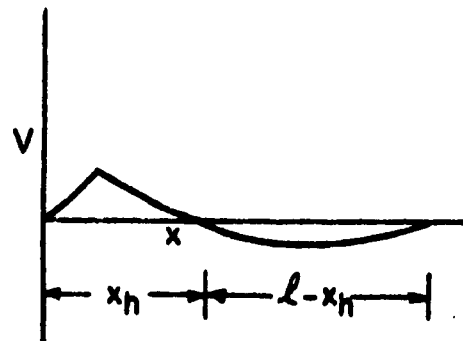


Fig. 8(b)

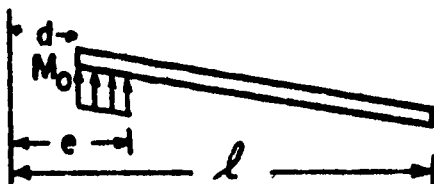


Fig. 9

Free body diagram of segment to right of central plastic region

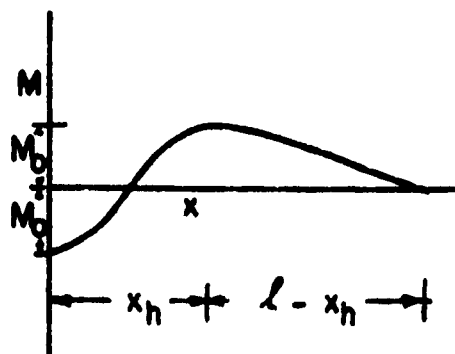


Fig. 8(c)

Diagrams of effective load, shear and bending moment in the third phase, with concentrated central hinge

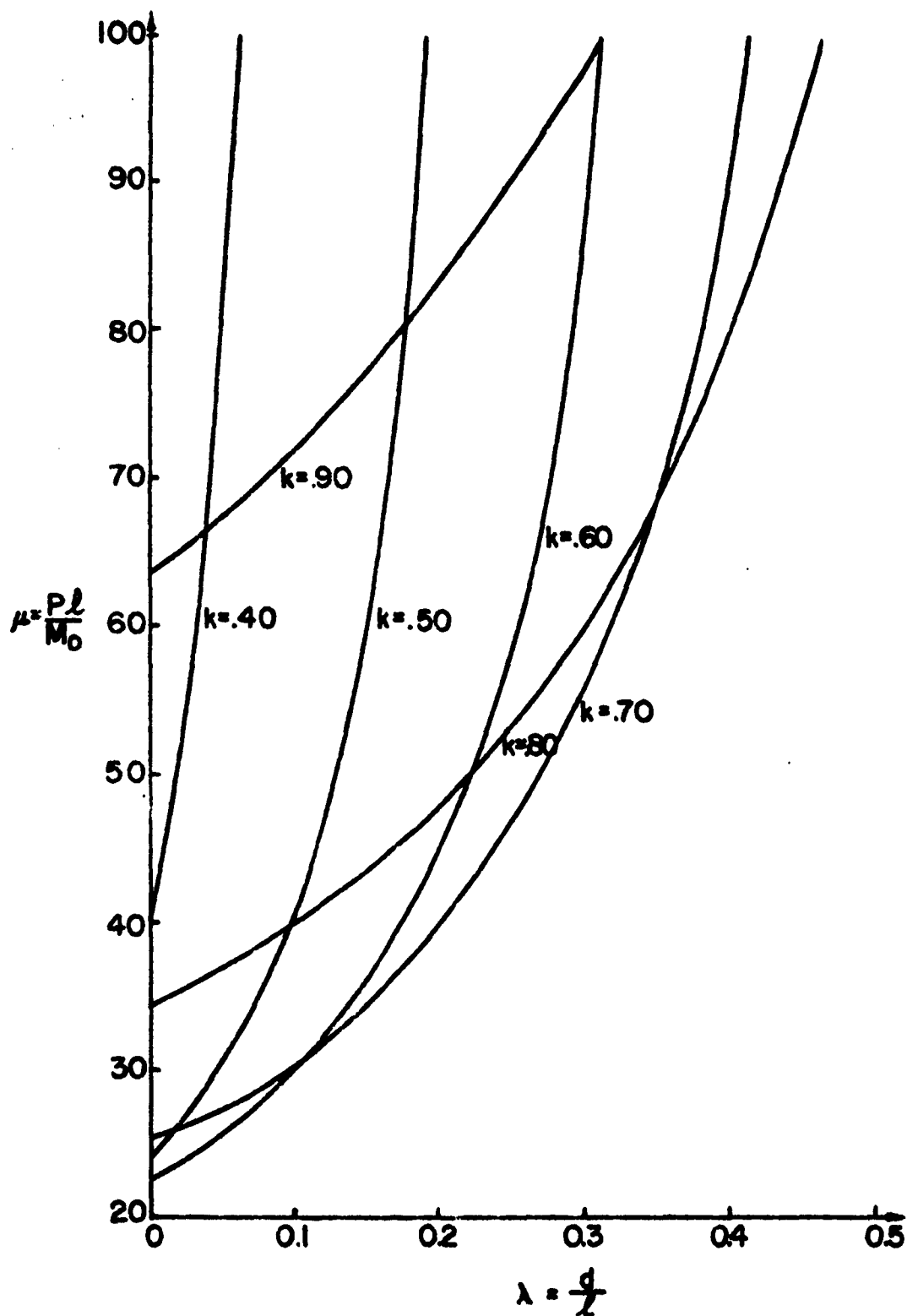


Fig. 10 Dependence of width  $d$  of central hinge on load  $P$  for various values of load distribution parameter  $k = e/l$

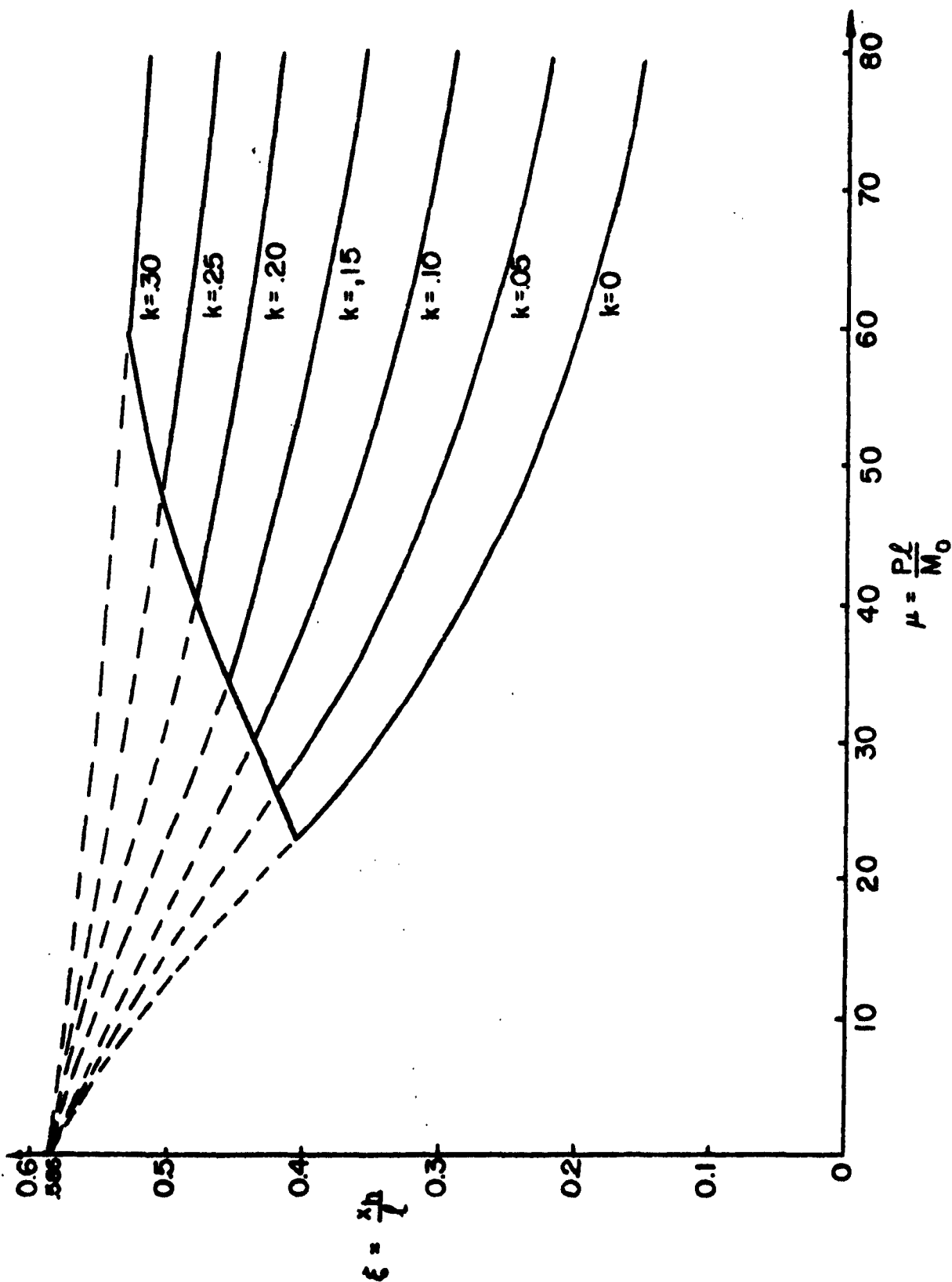


Fig. 11 Coordinate of lateral hinge  $x_h$  for various values of load distribution parameter  $k$ , for constant load  $P$ .

